Solids of Revolution

the integrals

Volume of $f(x) = \frac{x}{2}$

The area of a cross sectional disk is:

$$
\begin{aligned}
A(x) &= \pi f(x)^2 \\
&= \pi(\frac{x}{2})^2 \\
&= \pi\frac{x^2}{4}
\end{aligned}
$$

The volume of the solid formed by a series of these disks is:

$$
\begin{aligned}
V &= \pi\int_a^b \frac{x^2}{4} dx \\
&= \frac{\pi}{4}\int_a^b x^2 dx \\
&= \frac{\pi x^3}{12}\Big|_a^b
\end{aligned}
$$

We can check our math by selecting the limits $a = 2, b = 4$ and finding the volume of the resulting frustum using both the integration and the formula for the volume of a frustum.

$$
\begin{aligned}
V &= \frac{\pi x^3}{12}\Big|_2^4 \\
&= \frac{4^3\pi}{12} - \frac{2^3\pi}{12} \\
&= \frac{64\pi}{12} - \frac{8\pi}{12} \\
&= \frac{56\pi}{12} \\
&= \frac{14\pi}{3}
\end{aligned}
$$

and by the formula keeping in mind the function $f(x) = \frac{x}{2}$ gives the radii of the two ends of the frustum and the height is the difference in the limits $(R = \frac{4}{2} = 2, r = \frac{2}{2} = 1, \text{ and, } h = 4 - 2 = 2)$

$$\begin{aligned} V &= \frac{\pi h}{3}(R^2 + Rr + r^2) \\ &= \frac{2\pi}{3}(2^2 + 2(1) + 1^1) \\ &= \frac{2\pi}{3}(7) \\ &= \frac{14\pi}{3} \end{aligned}$$

Volume of $f(x) = \sin(x)$

The area of a cross sectional disk is:

$$\begin{aligned} A(x) &= \pi f(x)^2 \\ &= \pi \sin(x)^2 \\ &= \pi \sin^2(x) \end{aligned}$$

The volume of the solid formed by a series of these disks is:

$$\begin{aligned} V &= \int_a^b \pi \sin^2(x) dx \\ &= \pi \int_a^b \sin^2(x) dx \\ &= \pi \frac{2x - \sin(2x)}{4} \Big|_a^b \end{aligned}$$

The above result doesn't just fall out of the sky, using the reduction formula where $n = 2$

$$\begin{aligned} \int \sin^n(x) dx &= \frac{n-1}{n} \int \sin^{n-2}(x) dx - \frac{\cos(x)\sin^{n-1}(x)}{n} \\ &= \frac{2-1}{2} \int \sin^{2-2}(x) dx - \frac{\cos(x)\sin^{2-1}(x)}{2} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}\int sin^0(x)dx - \frac{\cos(x)\sin(x)}{2} \\ &= \frac{1}{2}\int 1dx - \frac{\cos(x)\sin(x)}{2} \\ &= \frac{x}{2} - \frac{\cos(x)\sin(x)}{2} \end{aligned}$$

using the identity $\sin(2x) = 2\sin(x)\cos(x)$ and solving for $\sin(x)\cos(x)$ gives

$$\sin(x)\cos(x) = \frac{\sin(2x)}{2}$$

substituting $\frac{\sin(2x)}{2}$ for $\sin(x)\cos(x)$ in the integration gives

$$\frac{x}{2} - \frac{\frac{\sin(2x)}{2}}{2}$$

applying the algebraic rule $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$ gives

$$\frac{x - \frac{\sin(2x)}{2}}{2}$$

which can be reduced using $\frac{\frac{b}{c}}{a} = \frac{b}{a(c)}$ to

$$\frac{2x - \sin(2x)}{4}$$